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STOCHASTIC LOCATION-ROUTING PROBLEM**

by

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Abstract

In this article, a stochastic location-routing problem is defined and cast as a two-stage model. In a first stage, the set of plants and a family of routes are determined; in a second stage, a recourse action is applied to adapt these routes to the actual set of customers to visit, once they are known. A two-phase heuristic is developed. An initial feasible solution is built by solving a sequence of subproblems, and an improvement phase is then applied. A lower bound based on bounding separately different parts of the cost of any feasible solution is also developed. Computational results are reported.

Keywords: stochastic location-routing problems.

Résumé

Dans cet article, on définit un problème stochastique mixte de localisation et de tournées de véhicules. Dans une première étape, on détermine un ensemble de localisations d'installations et de tournées de véhicules ; dans une deuxième étape, on applique une politique de recours pour adapter ces tournées à l'ensemble des clients devant effectivement être visités. On développe une heuristique en deux phases. On construit une solution initiale admissible par la résolution d'une suite de sous-problèmes et on applique ensuite une procédure d'amélioration. On calcule une borne inférieure sur divers éléments des coûts d'une solution admissible. On présente des résultats numériques.

Mots-clés : problèmes stochastiques mixtes de localisation et de construction de tournées.

1 Introduction

The problem under study combines two types of decision that arise most often in strategic planning, namely the location of facilities and the design of routes. Problems where decisions concerning multiple issues have to be made have been classically tackled by a succession of models that consider one issue at a time. Nowadays, however, major efforts are being devoted to the development of tools capable of handling different decision levels simultaneously. In the case of combined *Location-Routing Problems* (LRPs) several variants have already been studied. A classification of LRPs can be found in Laporte (1988) and a more recent literature review is provided in Min, Jayaraman, and Srivastava (1998). Although exact solutions have been proposed for some cases (Ghosh 1981; Laporte, Nobert, and Arpin 1986; Laporte, Nobert, and Taillefer 1988), LRPs are most often tackled by means of heuristics which usually address the location and the routing decisions separately.

A common assumption in mathematical programming is that all problem data are known in advance. In most cases, however, decisions have to be made before the realization of a random variable is known. A classical approach is to work with estimations of random data and solve the stochastic problem as in the deterministic case. However, it is often preferable to explicitly incorporate uncertainty in the models. The recent growth of the stochastic programming literature (see, e.g., Birge and Louveaux 1997) illustrates the increasing interest in such models.

We consider a stochastic LRP in which a set of potential customers is given but only a subset of them will require service after the *a priori* decision is made. Uncertainty is modeled by using a vector of independent Bernoulli random variables as the demand vector. In this specific *Stochastic Location-Routing Problem* (SLRP), capacities on the plants are given in terms of the number of customers each plant can actually serve, and vehicles (one for each open plant) are uncapacitated. Very few articles on SLRPs are known. Some examples are provided by Burness and White (1976), Harrison (1979), and Laporte, Louveaux, and Mercure (1989).

The SLRP considered in this paper is handled by means of a two-stage model where in the first stage one determines the set of open plants and *a priori* routes visiting each customer exactly once (see Figure 1a)). In the second stage, once the subset of customers requiring service

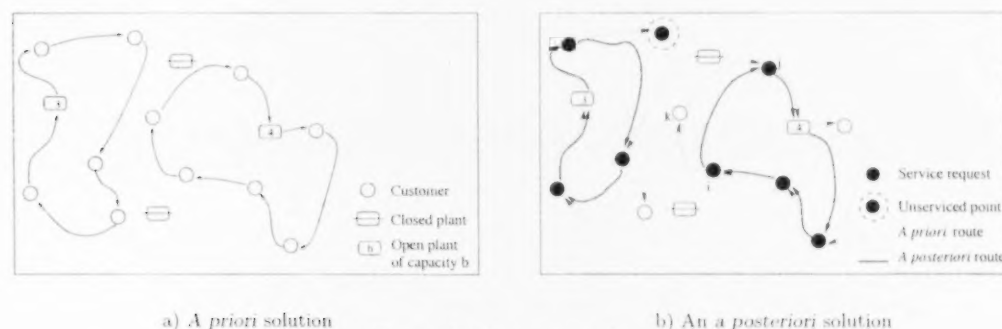


Figure 1: Recourse action

is known, the actual *a posteriori* routes have to be designed. We consider that the allocation of customers to the open plants remains unaltered. We also assume that when a plant is overloaded, i.e., the number of customers on its route requesting service exceeds its capacity, some of these customers are left unserved, which generates an additional cost. This additional cost can be interpreted as the penalty for losing a customer, or as the cost of acquiring external resources to provide service. Here, the set of unserved customers is randomly chosen contrary to what is done under classical policies in which the unserved customers are the last in their route or are those requiring the longest travel time. A reason for our choice of policy is to avoid prioritizing customers that have a privileged position in their *a priori* route. Once the set of customers to be actually serviced is decided, these are visited in the order defined by the *a priori* routes, as depicted in Figure 1b).

The goal in this SLRP is to minimize the total cost, defined as the sum of the fixed costs for opening the plants plus the expected cost of the recourse action. The cost of recourse is the sum of the expected penalties incurred for unserved customers and of the expected costs of the *a posteriori* routes. We further assume that the requests for customer service are identically distributed and independent of each other (*i.i.d.*). This will allow us to find an analytic expression of the recourse function, partially derived from the one presented in Jaillet (1988) for the *Probabilistic Traveling Salesman Problem* (PTSP).

The remainder of this article is organized as follows. A mathematical model is presented in Section 2, followed by a heuristic in Section 3, and by a lower bound in Section 4. Computational experiments and conclusions are presented in the last two sections.

2 Mathematical model

In this section we present a mathematical model for the SLRP under study. The notation is first introduced. We then present the global model for the SLRP. The two last subsections are devoted to the exact and approximate computation of the recourse function value.

2.1 Notation

An instance of the SLRP under consideration is defined by:

- a set I of potential facility locations;
- a set J of customer locations;
- non-negative travel costs c_{ij} between pairs of customers or between customers and plants;
- fixed costs f_i for opening plants at sites $i \in I$;
- capacities b_i , for each potential plant location $i \in I$;
- a parameter p defining the probability distribution of the service request for each customer and its complement $q = 1 - p$; we assume that customers make service requests independently, all with the same probability. Thus, the demands of the customers $(\xi_j, j \in J)$ can be modeled by means of independent Bernoulli random variables with a common parameter p ;
- a penalty cost g incurred when the request for service of a customer is disregarded. We assume that this penalty is the same for all customers.

In what follows, feasible *a priori* solutions will be characterized by:

- binary variables $y_i, i \in I$ indicating whether plant i is open or not;
- for each $i \in I$ with $y_i = 1$:
 - $J(i) \subseteq J$, the set of customers allocated to plant i ;
 - $n_i = |J(i)|$, the number of customers allocated to plant i ;
 - x_i , the incidence vector of the arcs in the *a priori* route associated with plant i ;
 - $(j_1, \dots, j_{n_i})$, a permutation of the indices of $J(i)$ giving the order in which customers are visited in route x_i .

Given a realization ξ of the demand vector, for each $i \in I$ with $y_i = 1$, we define:

- η_i , the total demand allocated to plant i ,

$$\eta_i = \sum_{j \in J(i)} \xi_j.$$

Because service requests are *i.i.d.* Bernoulli random variables, it follows that

$$\eta_i \sim \text{Binomial}(n_i, p).$$

- $J'(i)$, the set of customers actually visited in the *a posteriori* route associated with plant i . We can distinguish two cases in the definition of $J'(i)$: if $\eta_i \leq b_i$, then $J'(i) = \{j \in J(i) : \xi_j = 1\}$; otherwise, if $\eta_i > b_i$, then $J'(i)$ is a random subset of $\{j \in J(i) : \xi_j = 1\}$ of cardinality b_i .
- x'_i , the incidence vector of the arcs in the route obtained from x_i by skipping the customers not in $J'(i)$.

2.2 Formulation

Using the above notation we can model the objective function as:

$$\sum_{i \in I} f_i y_i + Q(x),$$

where the recourse function $Q(x)$ is defined as

$$Q(x) = \mathbb{E}_\xi \left[\sum_{i|y_i=1} g(\eta_i - b_i)^+ + \mathbb{E}_{J'(i)} \left[\sum_{i|y_i=1} c x'_i \right] \right], \quad (1)$$

and $(\cdot)^+ = \max\{\cdot, 0\}$. Using linearity of the expectation function, (1) reduces to

$$Q(x) = S(x) + \sum_{i|y_i=1} T^i(x),$$

where

$$S(x) = g \sum_{i|y_i=1} \mathbb{E}_\xi[(\eta_i - b_i)^+]$$

are the expected penalties, and

$$T^i(x) = \mathbb{E}_\xi [\mathbb{E}_{J'(i)}[c x'_i]]$$

is the expected cost of route $i \in I$.

Note that $S(x)$ fits exactly the structure of the recourse function associated with a stochastic problem with simple recourse, extensively studied in the literature (see, for instance, Birge and Louveaux 1997). In contrast, the functions $T^i(x)$ do not have a simple structure.

2.3 Exact computation of the recourse function

To determine an analytical expression for $T^i(x)$, we must compute the probabilities of actually using each of the arcs connecting any pair of vertices in $J(i) \cup \{i\}$. Given the assumption that service requests are *i.i.d.* and the definition of the recourse action, the probability of using a certain arc does not depend on its endpoints, but only on the number of customers skipped, according to the order defined by the *a priori* solution x . For arcs connecting two customers having ℓ intermediate customers ($0 \leq \ell \leq n_i - 2$) in the *a priori* route corresponding to plant i , standard probability computations (see the appendix for details) yield the expression

$$P_\ell = p^2 \left[\sum_{k=2}^{k_1} \binom{n_i - \ell - 2}{k-2} p^{k-2} q^{n_i-k} + \sum_{k=b_i+1}^{n_i} p^{k-2} q^{n_i-k} \sum_{t=t_0}^{\ell_1} \binom{\ell}{t} \binom{n_i - \ell - 2}{k-t-2} \binom{k-t-2}{b_i-2} / \binom{k}{b_i} \right]. \quad (2)$$

The factor p^2 in (2) corresponds to the probability that the endpoints of the considered arc have demand, whereas the second factor corresponds to the conditional probability of actually visiting them and servicing them consecutively, given that they have positive demand.

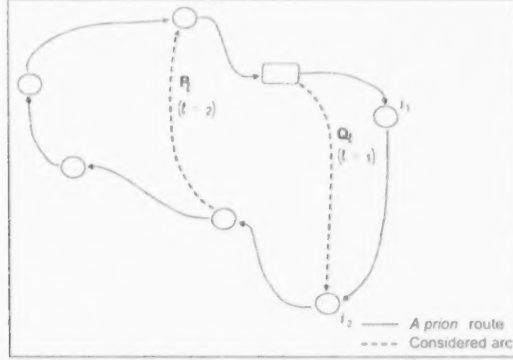


Figure 2: Probabilities associated with the arcs

Similarly, we can compute the probabilities associated with arcs incident to a plant, skipping again ℓ customers:

$$R_\ell = p \left[\sum_{k=1}^{k_1} \binom{n_i - \ell - 1}{k-1} p^{k-1} q^{n_i-k} + \sum_{k=b_i+1}^{n_i} p^{k-1} q^{n_i-k} \sum_{t=t_0}^{t_1} \binom{\ell}{t} \binom{n_i - \ell - 1}{k-t-1} \binom{k-t-1}{b_i-1} / \binom{k}{b_i} \right] \quad (3)$$

In both (2) and (3), $k_1 = \min\{b_i, n_i - \ell\}$, $t_0 = \max\{k - n + \ell, 0\}$, and $t_1 = \min\{k - b_i, \ell\}$. Combining (2) and (3) yields the recourse function:

$$T^i(x) = \sum_{h=1}^{n_i} R_{h-1} c_{ij_h} + \sum_{h=1}^{n_i-1} \sum_{t=h+1}^{n_i} P_{t-h-1} c_{j_h j_t} + \sum_{h=1}^{n_i} R_{n_i-h} c_{j_h i}. \quad (4)$$

2.4 Approximate computation of the recourse function

Evaluating the function T^i described in (4) can be time consuming, given the complexity of the expressions (2) and (3) which contain many binomial coefficients. For this reason we will use an approximation of T^i which can be evaluated much faster. Approximate objective functions in algorithms requiring an important number of evaluations have already been employed successfully in stochastic programming (see, for example, Gendreau, Laporte, and Séguin 1996). In our situation the difficulty in evaluating the cost of the recourse is largely due to the fact that visits to different customers in the same *a priori* route are not independent because of the capacity constraint of the plant. To overcome this source of complexity we propose to build an approximation $\tilde{T}^i$ of the recourse function where these probabilities are substituted by those that would result if services to customers were independent. In fact, this is the case when $n_i \leq b_i$.

Let $\bar{p}$ be the probability of visiting a given customer, computed as p times the conditional probability of visiting that customer given that it has demand. If the customer is assigned *a priori* to plant i , this conditional probability is a sum of probabilities over all possible values $\eta_i = k \in \{1, \dots, n_i\}$ of the number of customers with demand in its route. Straightforward computations give:

$$\bar{p} = p \left(\sum_{k=1}^{b_i} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} + \sum_{k=b_i+1}^{n_i} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} b_i/k \right).$$

Substituting the probabilities P_ℓ and R_ℓ by

$$\tilde{P}_\ell = \bar{p}^2 (1-\bar{p})^\ell \quad \text{and} \quad \tilde{R}_\ell = \bar{p} (1-\bar{p})^\ell$$

in (4), we obtain the following approximation of T^i :

$$\tilde{T}^i = \sum_{k=1}^{n_i} \bar{p} (1-\bar{p})^{k-1} c_{ijk} + \sum_{k=1}^{n_i-1} \sum_{t=k+1}^{n_i} \bar{p}^2 (1-\bar{p})^{t-k-1} c_{jkt} + \sum_{k=1}^{n_i} \bar{p} (1-\bar{p})^{n_i-k} c_{jki}.$$

Note that this expression coincides with the objective function proposed in Jaillet (1988) for the PTSP, for the particular case where the probability of presence of all customers is the same. In what follows, we will refer to the approximation of the recourse function as $\tilde{Q}(x)$.

3 Heuristic for the Stochastic Location-Routing Problem

We propose a two-phase heuristic for the SLRP. An initial solution generated in the construction phase is iteratively improved in a local search phase.

3.1 Construction phase

In the construction phase an initial solution is gradually built by splitting the problem into a sequence of three small subproblems: 1) select the set of plants to open, 2) determine the allocation of customers to open plants, and 3) design a route for each open plant.

3.1.1 Selection of the set of open plants

The set O of plants to open is chosen in such a way that the overall capacity is sufficient to satisfy all the service requests within a prespecified probability α , that is:

$$\mathbb{P} \left[\sum_{j \in J} \xi_j \leq \sum_{i \in O} b_i \right] \geq \alpha. \quad (5)$$

The smallest overall capacity b_α satisfying (5) is easily derived from the binomial probability distribution of the total number of service requests. Once b_α is computed, O is determined by solving the following *Knapsack Problem*:

$$(KP_\alpha) \quad \text{minimize} \quad \sum_{i \in I} f_i y_i \quad (6)$$

$$\text{subject to} \quad \sum_{i \in I} b_i y_i \geq b_\alpha \quad (7)$$

$$y_i \in \{0, 1\} \quad \forall i \in I. \quad (8)$$

Observe that, by construction of b_{α} , constraints (5) and (7) are equivalent. Thus, the set $O = \{i \in I : y_i = 1 \text{ in the solution of } KP_{\alpha}\}$ has minimal cost among those satisfying (5).

3.1.2 Allocation of customers to the open plants

Once the set of plants is fixed, an appropriate allocation of customers to open plants has to be determined. To this end, we solve the *Stochastic Generalised Assignment Problem* (SGAP) where the cost function is given by the distances between plants and customers. Different heuristics for the SGAP have been proposed in Albareda-Sambola and Fernández (2000). To find a suitable allocation, we use Heuristic B of that paper. Since customer reassignments are not allowed in the SLRP, we simply set all reassignment costs to a value higher than the penalty for not servicing them when they have a demand.

3.1.3 Construction of the set of *a priori* routes

Starting from empty routes, customers are successively inserted in the route of the plant they have been assigned to, using a nearest insertion criterion with respect to the deterministic arc costs. After all customers have been inserted in their route, successive interchanges of two arcs within the same route are performed until all routes are 2-opt with respect to the objective function $\bar{T}$.

3.2 Improvement phase

Once an initial solution has been obtained in the construction phase, it is improved by local search. All moves are performed using the nearest insertion criterion with respect to $\bar{T}$. We use four neighborhoods:

- $N_1(x)$: *Reassignment of one customer*. $N_1(x)$ contains all solutions that can be derived from x by taking a customer from its route and inserting it in a different one.
- $N_2(x)$: *Interchange of two customers*. $N_2(x)$ contains all solutions that can be obtained from x by taking a pair of customers j_1 and j_2 belonging to different routes i_1 and i_2 , respectively, removing them from their routes, and inserting customer j_1 in the route of plant i_2 and j_2 in the route of plant i_1 .
- $N_3(x)$: *Plant interchange*. Solutions in $N_3(x)$ are those that can be obtained from x by closing an open plant i_1 , opening a closed plant i_2 , and assigning the route of i_1 to i_2 .
- $N_{2-opt}(x)$: *Arc interchange*. Solutions in $N_{2-opt}(x)$ are obtained from x by interchanging pairs of arcs in the same route.

The heuristic is outlined in Figure 3, where the order of exploration of the different neighborhoods is shown.

4 Lower bound for the Stochastic Location-Routing Problem

To obtain valid lower bounds for the SLRP we have split the objective function into two parts and we have derived a separate bound for each of them. These two parts are 1) the cost associated with plants (fixed costs plus expected penalties), and 2) the cost of the actual routes.

SLRP Heuristic

```

{Constructive Phase}
Select  $\alpha$ , determine  $b_\alpha$  and solve  $(KP_\alpha) \rightarrow O$ .
Allocate customers to open plants using Heuristic B  $\rightarrow \{J(i) : i \in O\}$ .
for ( $i \in O$ ) do
  for ( $j \in J(i)$ ) do
    Insert  $j$  in the route of plant  $i$ .
  end for
  Explore  $N_{2\text{-opt}}(x)$ .
end for
{Improving Phase}
StopCriterion  $\leftarrow$  false.
while (not StopCriterion) do
  repeat
    Explore  $N_2(x)$  and update  $x$ .
    Explore  $N_1(x)$  and update  $x$ .
  until (no improving moves found)
  repeat
    Explore  $N_{2\text{-opt}}(x)$  and update  $x$ .
  until (no improving moves found)
  Explore  $N_3(x)$  and update  $x$ .
  Update StopCriterion.
end while

```

Figure 3: Outline of the SLRP heuristic procedure

4.1 Fixed costs for opening plants plus expected penalties for unserved customers: $\underline{z}_{SKP}$

It is important to bound the expected penalty together with the fixed costs for opening plants because these two terms are conflicting. Opening a plant increases the fixed costs but decreases the penalties. The expected penalty associated with any solution with a set O of open plants, will not be smaller than

$$\mathbb{E}_\xi \left[g \left(\sum_{j \in J} \xi_j - \sum_{i \in O} b_i \right)^+ \right],$$

independently of the assignment of customers to plants. A lower bound on the sum of fixed costs plus expected penalties is therefore given by the solution of the following *Stochastic Knapsack Problem*.

$$\begin{aligned} \underline{z}_{SKP} = \text{minimize} \quad & \sum_{i \in I} f_i y_i + \mathbb{E}_\xi \left[g \left(\sum_{j \in J} \xi_j - \sum_{i \in I} y_i b_i \right)^+ \right] \\ \text{subject to} \quad & y_i \in \{0, 1\}, \quad i \in I. \end{aligned}$$

This problem is a two-stage problem with simple recourse. It can be seen that a deterministic equivalent of this problem is

$$\begin{aligned}
& \text{minimize} && \sum_{i \in I} f_i y_i + \sum_{s=0}^{n+1} \delta^s u^s \\
& \text{subject to} && \sum_{i \in I} b_i y_i - \sum_{s=0}^{n+1} u^s = np \\
& && u^0 \leq 1 - np \\
& && 0 \leq u^{l+1} \leq 1 && l = 0, \dots, n \\
& && y_i \in \{0, 1\} && i \in I,
\end{aligned}$$

where $\delta^0 = -g$, and $\delta^{l+1} = g \left(-1 + \mathbb{P} \left[\sum_{j \in I} \xi_j \leq l \right] \right)$, $l = 0, \dots, n$.

This is a mixed integer programming problem which can be efficiently solved using any general purpose software.

4.2 Expected cost of the *a posteriori* routes

To compute a lower bound on the cost of the *a posteriori* routes we use a procedure similar to that proposed by Laporte, Louveaux, and Mercure (1994) for the PTSP. That is, we compute a lower bound on the cost $\bar{z}_{RS}$ of the *a priori* routes and we subtract from it an upper bound on the expected savings $\bar{z}_{TSP}$ for skipping customers.

4.2.1 Lower bound $\bar{z}_{TSP}$ on the cost of the *a priori* route

To bound the cost of the *a priori* route we use the solution to a *Traveling Salesman Problem* (TSP) defined as follows. We consider a complete directed graph with vertex set $V = I \cup I' \cup J$, where I' is a copy of the set of plants. Vertex i represents the plant as a starting point of a route, and i' represents the same plant, as an ending point of the route. We define the following arc costs:

$$\bar{c}_e = \begin{cases} c_e & \text{if } e \in J \times J, e \in I \times J \text{ or } e \in J \times I' \\ 0 & \text{if } e \in I' \times I \text{ or } e = (i, i'), i \in I \\ \infty & \text{otherwise.} \end{cases}$$

With these costs, vertices representing plants in the optimal TSP solution will be connected only in two cases. An arc from an ending-plant vertex to a starting-plant vertex corresponds to ending a route in one plant and starting another route in the next plant, while an arc from a starting-plant vertex to its corresponding ending-plant vertex represents an empty route. Figure 4 illustrates how a feasible solution for this TSP can be derived from any *a priori* set of routes for the SLRP. For this TSP we use a model based on the one proposed by Miller, Tucker, and Zemlin (1960) and strengthened by Desrochers and Laporte (1991):

$$\text{minimize } \sum_{e \in A} c_e x_e \quad (9)$$

$$\text{subject to } \sum_{e \in \delta^-(j)} x_e = 1 \quad i \in V \quad (10)$$

$$\sum_{e \in \delta^+(j)} x_e = 1 \quad i \in V \quad (11)$$

$$u_i - u_j + (\bar{n} - 1)x_{ij} + (\bar{n} - 3)x_{ji} \leq \bar{n} - 2 \quad i, j > 1, i \neq j \quad (12)$$

$$x_e \in \{0, 1\} \quad e \in A \quad (13)$$

$$1 \leq u_i \leq \bar{n} - 1 \quad i > 1, \quad (14)$$

where $\bar{n} = |V| = 2m + n$, A is the set of arcs of the graph and, given a vertex j , $\delta^-(j)$ (resp. $\delta^+(j)$) is the set of arcs entering (resp. leaving) j .

In this model, variables x_e indicate whether arc e is used in the solution, and each variable u_i gives the position of vertex i in the tour. In order to tighten the bound provided by this model, some side constraints have been added.

- 1) Different TSP solutions can be derived from the same *a priori* SLRP solution, depending on the order in which the plants are considered. Without loss of generality, this difficulty can be avoided by fixing the order of the plants through the constraints

$$u_i - u_{i-1} \geq 2, \quad i \in I \setminus \{0\}. \quad (15)$$

- 2) Using the order specified by constraints (15), it is easy to force each route to end at the plant where it started. We only need to impose

$$u_i - u_{(i-1)'} \geq 1, \quad i \in I \setminus \{0\}. \quad (16)$$

We denote by $\underline{z}_{TSP}$ the optimal value of (9)-(14), reinforced by constraints (15) and (16).

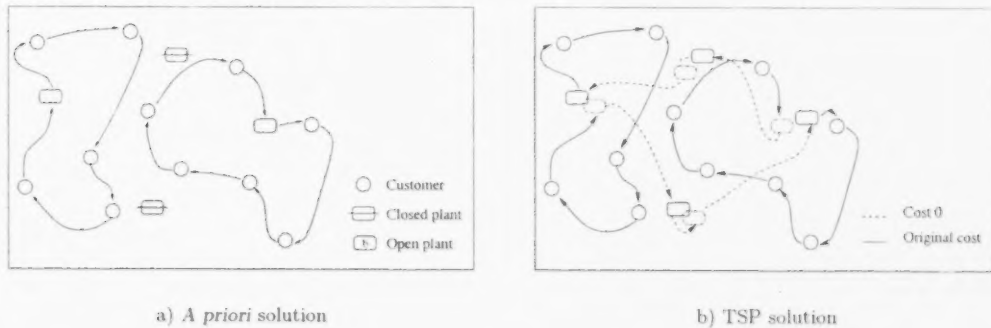


Figure 4: Bound on the *a priori* route cost, $\underline{z}_{TSP}$

4.2.2 Upper bound $\bar{z}_{RS}$, on the expected savings for skipping customers

To bound the expected savings in the *a posteriori* routes we use an extension of the model proposed in Laporte, Louveaux, and Mercure (1994) for the PTSP. The idea behind this model is to find a suitable family of compatible shortcuts having a maximal total value. A shortcut represents the substitution of a path (i, k, j) in the *a priori* solution by the arc (i, j) (see Figure 1b). In our case, for each customer k we define sets of triplets for all possible shortcuts for k as follows:

- 1) $A(k) = \{(i, k, i) : i \in I\};$
- 2) $B(k) = \{(i, k, j) : i \in I, j \in J, j \neq k\};$
- 3) $C(k) = \{(j_1, k, j_2) : j_1, j_2 \in J, j_1, j_2 \neq k, j_1 \neq j_2\},$ and
- 4) $D(k) = A(k) \cup B(k) \cup C(k).$

For each triplet (i, k, j) we define a binary variable y_{ikj} . In addition, we include binary variables w_i for $i \in I$ to indicate whether plant i is open or not. The coefficients in the objective function correspond to the values of the shortcuts. Given a triplet (i, k, j) , the corresponding coefficient is given by

$$d_{ikj} = q(c_{ik} + c_{kj} - c_{ij}),$$

where indices i, j correspond to customers or plants. Note that if $i = j \in I$ (type A variable), then the last term c_{ij} is zero. Using the above notation, the resulting model is as follows:

$$\text{maximize } \sum_{k \in J} \sum_{(i, k, j) \in D(k)} d_{ikj} y_{ikj} \quad (17)$$

$$\text{subject to } \sum_{(i, k, j) \in D(k)} y_{ikj} = 1 \quad k \in J \quad (18)$$

$$\sum_{k \in J} y_{ikl} \leq 1 \quad i \in I \quad (19)$$

$$\sum_{k \in J \setminus \{j\}} y_{ikj} \leq 2 \quad i \in I, j \in J \quad (20)$$

$$\sum_{k \in J \setminus \{j_1, j_2\}} y_{j_1 k j_2} \leq 1 \quad j_1 \neq j_2 \in J \quad (21)$$

$$\sum_{k \in J \setminus \{j\}} \sum_{i \in I} y_{ikj} + \sum_{\substack{k, j_2 \in J \\ k \neq j_2}} y_{j k j_2} \leq 2 \quad j \in J \quad (22)$$

$$2 \sum_{k \in J} y_{ikl} + \sum_{k, j \in J} y_{ikj} \leq 2w_i \quad i \in I \quad (23)$$

$$\sum_{i \in I} f_i w_i \leq n p g \quad (24)$$

$$2 \sum_{i \in I} y_{iji} + \sum_{i \in I} \sum_{\substack{k \in J \\ k \neq j}} y_{ikj} + \sum_{\substack{k, j_2 \in J \\ j \neq k \neq j_2 \neq j}} y_{j k j_2} \leq 2 \quad j \in J \quad (25)$$

$$y_{ikj}, w_i \in \{0, 1\} \quad k \in J, (i, k, j) \in D(k), i \in I. \quad (26)$$

In this model, constraints (18) ensure that exactly one shortcut is selected for each customer, and constraints (19)-(22) are used to control the use of any vertex as an endpoint of a shortcut. In general, any vertex can appear as the endpoint of two different shortcuts, with the exception of plants with shortcuts of type A, which can appear only once. Constraints (23) prevent the association of shortcuts with unopened plants. Constraint (24) sets an upper bound for the total fixed costs of the open plants. The right-hand side upg is the cost of a solution that does not open any plant. Since it is equal to the cost of a feasible solution for the SLRP, it is an upper bound of the cost of the optimal solution and, in particular, of the fixed opening costs associated with it. Finally, constraints (25) prevent customers associated with a shortcut of type A from being endpoints of any other shortcut. The optimal solution to this model provides an upper bound z_{HS} of the expected savings for skipping customers in any set of feasible routes.

5 Computational results

The heuristic of Section 3 as well as the lower bound of Section 4 were tested on a series of instances. In this section we present the different experiments that were carried out and the associated results. Subsection 5.1 describes the instance generation process. In Subsection 5.2 the results obtained by the heuristic and the lower bound for the generated instances are reported while Subsection 5.3 describes an additional experiment carried out to test the quality of the approximation of the recourse function.

5.1 Instance generation

The instances we use are derived from those in <http://troubadix.unisg.ch/klose/problems/problems.html#TSCFLP> for the *Two-Stage Capacitated Facility Location Routing*. There are five groups consisting of 25 instances each. Groups SS1, SS2, and SS3 contain instances with 5 plants and 10, 20 and 30 customers, respectively, while instances with 10 plants and, respectively 20 and 30 customers make up groups SM2 and SM3. Within each group, instances were classified according to the ratio between the number of customers and the total capacity. This ratio takes the values 4.5, 4, 2.5, 2 and 1.5, and the resulting subgroups are labelled *a, b, c, d* and *e*. When the existing instance had more plants or customers than needed, the data corresponding to the exceeding plants or customers were ignored. Traveling costs were set equal to the Euclidean distances between points, and capacities were scaled in order to meet the prespecified ratios. Also, the fixed costs for opening the plants (depots in the original instances) were divided by 20 to obtain instances where operating and routing costs are on a similar scale. The penalties for not servicing a customer were determined according to the relationship

$$g = 2 \max\{c_{ij} : i \in I, j \in J\} + \max\{f_i/c_i : i \in I\}.$$

Finally, for each set of data, three different values of the probability of demand were considered: 0.2, 0.5 and 0.8.

5.2 Experimental results

In this subsection we display the results obtained by the heuristic and by the lower bound procedure. We first study the quality of the heuristic solutions by comparing the solution determined in the construction phase with that obtained after the improvement phase. We then study the behavior of the different parts of the lower bound.

For a given instance, let z_{best} be the value of the objective function of the solution obtained by the heuristic, and z_{ini} the value of the solution found in the construction phase. Figure 5 shows the percent deviation $100(z_{\text{ini}} - z_{\text{best}})/z_{\text{best}}$. Each bar represents the average deviation obtained within one subset of five instances, for a fixed demand probability. As can be seen, the most significant deviations correspond to high values of the demand probability. Nevertheless, even in the case of $p = 0.2$, significant deviations of up to 57.51% were obtained. The same figure shows the effect of instance size on the behavior of the local search procedure. The positive correlation between this behavior and instance size is due to the increase in the number of feasible solutions.

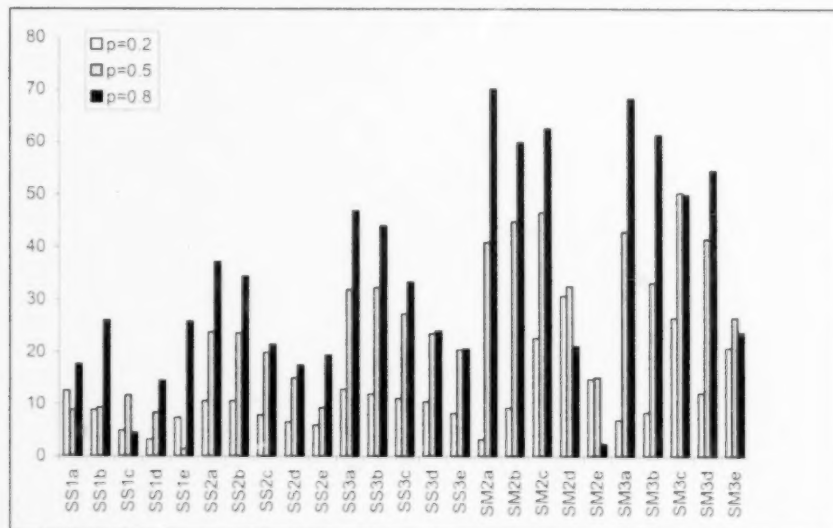


Figure 5: Percent improvements of the Local Search

We next report on the results obtained with the lower bounding procedure. In Table 1 we compare the sum of the two terms *FOC* (fixed opening costs) and *EP* (expected penalties) of the objective function with its lower bound $\underline{z}_{SKP}$. Each entry of this table corresponds to the average, for the instances of same dimension for a given probability, of the percent gap $100(FOC + EP - \underline{z}_{SKP})/\underline{z}_{SKP}$. This gap shows how close this part of the bound is to the corresponding value of the feasible solutions found. Therefore, a small value of the gap identifies instances where the lower bound was tight and the value of $FOC + EP$ of the solution found was small.

The results obtained for $p = 0.2$ are very satisfactory since the average gap for each dimension is under 1%. However, this good behavior is not maintained when the probability of demand increases. We conjecture that this is due to the aggregation of the capacities of the open plants in this lower bound. Since the probability of incurring penalties for leaving customers unserved increases with the probability of demand, the effect of this aggregation also becomes stronger. We can observe, however, that instance size does not have any influence in the quality of the bounds.

Table 1: Average percent gaps for fixed costs plus expected penalties

Group	$p = 0.2$	$p = 0.5$	$p = 0.8$
SS1	0.837	14.490	20.640
SS2	0.038	9.967	12.761
SS3	0.576	7.032	16.404
SM2	0.531	16.203	13.846
SM3	0.485	26.958	16.099

Unfortunately, the lower bound value $\bar{z}_{TSP} - \bar{z}_{RS}$ on the expected routing cost is unsatisfactory. We attribute this behavior to the following three factors: 1) As stated in Powell, Jaillet, and Odoni (1995), the solution of a deterministic TSP can be arbitrarily bad as a solution for the PTSP. To illustrate this, we compare in Figure 6 the routes obtained by the heuristic with the routes defined by the solution of the deterministic asymmetric TSP providing the lower bound. In Figure 6a) we show their *a priori* costs whereas the expected *a posteriori* costs are depicted in Figure 6b). In each case, only the instances in groups SS1a and SS2a, with $p = 0.2$ were considered. It can be observed that in some of the instances the two costs of the same

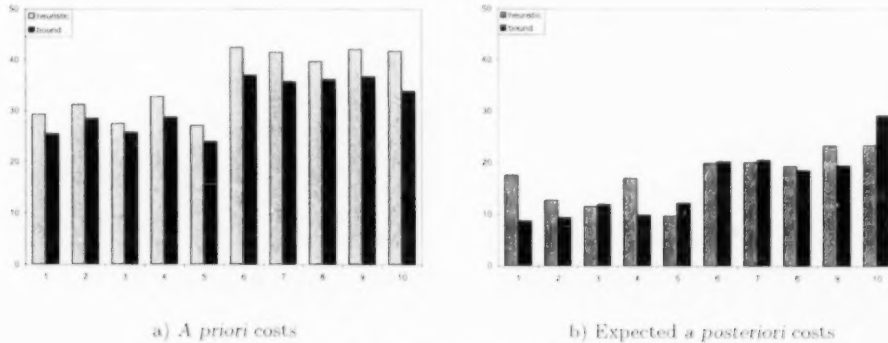


Figure 6: Route costs

set of routes compare differently. For example, in the fifth instance, the set of routes found by the heuristic is better than the one obtained with the lower bounding procedure, in terms of their expected costs. However, in terms of their *a priori* costs, the set of routes of the bounding procedure are better. 2) Routes found by the lower bounding procedure do not take into account plant capacities and whether the plants are open or not. This explains why, in some cases, the expected cost of the routes found by the bounding procedure is smaller than the expected cost of the routes found by the heuristic. 3) Finally, in the computation of $\bar{z}_{RS}$, the saving associated with triplet (i, k, j) is considered, although it is not likely that two customers that are far apart from each other will belong to the same *a priori* route. In practice, triplets involving distant vertices tend to be selected in the solution of problem (17)-(26) and correspond to large savings that are not likely to be achieved in the *a posteriori* solution.

5.3 Quality of the approximation of $Q(x)$

We conclude this section with an analysis of the quality of the function $\tilde{Q}(x)$ as an approximation of $Q(x)$. It is worth noting that each evaluation of $Q(x)$ requires the computation of some of the probabilities P_ℓ and R_ℓ . In general, this would imply a considerable increase in the computational costs. However, the small sizes of our instances enable the initial computation and storage of the probabilities P_ℓ and R_ℓ . Thus, it is possible to avoid computing these repeatedly during the process. We have implemented a version of the heuristic that proceeds in this way and computes $Q(x)$ exactly. In what follows, we will denote by EXACT this version of the heuristic and by APPR the version using the approximation. In most of the cases both versions gave the same solution although in several one of the versions outperforms the other. Table 2 shows the number of instances falling in each category, for different values of p . The results

Table 2: Comparison of APPR with EXACT

	Number of instances for which		
	EXACT is better	EXACT and APPR are the same	APPR is better
$p=0.2$	43	80	2
$p=0.5$	25	80	20
$p=0.8$	47	54	24

in this table are satisfactory: in more than half of the cases using the approximate objective function did not have any effect on the solution value, and the number of instances where APPR gave a better solution is large. In Table 3 we show the average percent deviations of the value of the solutions obtained with APPR from those obtained with EXACT. Each column corresponds

Table 3: Average percent deviations of APPR with respect to EXACT

	$p=0.2$	$p=0.5$	$p=0.8$
SS1	0.031	0.070	-1.382
SS2	0.000	-0.032	0.356
SS3	0.183	-0.023	0.112
SM2	22.005	2.017	3.864
SM3	20.428	2.936	-5.860

to a value of the probability of demand and rows are related to different instance sizes. It can be observed that in general, the use of the approximation of the objective function does not affect to the quality of the final solution, except for five instances responsible for the high values in the last two rows of column " $p = 0.2$ ". The results of Table 3 can be further appreciated in Figure 7 where we have computed the average percent improvements of the objective function value with respect to the initial solution found. Averages are computed over all instances of the same dimension with the same probability of demand. These values are shown for EXACT (in

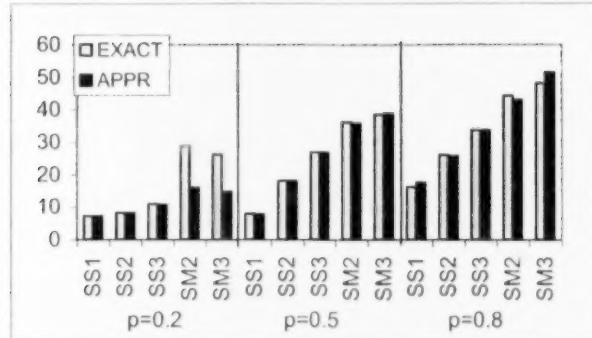


Figure 7: Comparison of the improvements obtained with EXACT and APPR

light grey) and for APPR (in black). For a fair comparison, given that both versions start from the same solution, the improvements were computed relative to the first solution. Again we can observe that both versions behave similarly in most of the groups, with the exception of SM2 and SM3 for $p = 0.2$.

The CPU times required by the two versions are comparable. A large proportion of instances, could be solved within one second for either version. Despite the fact that probabilities P_ℓ and R_ℓ are not computed repeatedly in EXACT, the proportion of instances where APPR was faster is very large. The reason is that in those instances, the local search performed by EXACT required more iterations before finding a local optimum. We can conclude that the approximation we have proposed for the recourse function is appropriate in this context. On the one hand, solutions obtained by APPR were most often similar in terms of quality. On the other hand, we note that our implementation of EXACT cannot be extended to larger instances, since then the storage requirement for the probabilities would be excessive.

6 Summary and conclusions

We have introduced a stochastic location-routing problem in which the presence or absence of service request of the customers is not known *a priori*. Customer demands were modeled as Bernoulli random variables and a two-stage model was proposed. In this model the set of plants to open and a set of *a priori* routes are determined at the first stage and, once the values of the demands become available, the routes are modified by skipping some of the customers. When the set of customers with demand assigned to an open plant has a total demand exceeding the plant capacity, a subset of customers to be served is randomly selected. Penalties are incurred for unserved customers.

We have presented a two-phase heuristic to solve this SLRP. It consists of a construction phase followed by local search. To reduce the computational effort during the improvement phase, we have used an approximation of the recourse function. We have also proposed a lower bound, obtained by bounding separately different parts of the objective function.

The quality of the heuristic and of the lower bound was assessed in a series of computational experiments. The effect of using the approximation of the recourse function was also studied.

Both the quality of the solutions provided by the heuristic and the associated computational effort required are excellent. In contrast, some elements of the lower bound are rather weak. Explanations are provided for this behavior. Our experiment relative to the quality of the approximation of the recourse function confirm the suitability of our approach.

Appendix. Computation of probabilities P_ℓ

We now provide the computation of the probabilities P_ℓ described in Section 2. Probabilities R_ℓ can be computed analogously.

Given an *a priori* tour associated with plant i , having n_i customers, and two customers j_1, j_2 belonging to this route, separated by ℓ intermediate customers ($\ell \leq n_i - 2$), the probability of using the arc (j_1, j_2) is equal to the probability p^2 that the endpoints have a demand, multiplied by the conditional probability of actually visiting them and servicing them consecutively, given that they have positive demand. This conditional probability can be decomposed into the sum of different probabilities obtained for different values of η_i , the number of customers in the route having demand.

The conditional probability that exactly k customers that are not located between j_1 and j_2 in the *a priori* route have a demand, given that j_1 and j_2 have a demand is:

$$p_k^- = \binom{n_i - 2}{k - 2} p^{k-2} q^{n_i - k} \binom{n_i - \ell - 2}{k - 2} / \binom{n_i - 2}{k - 2} = \binom{n_i - \ell - 2}{k - 2} p^{k-2} q^{n_i - k}. \quad (27)$$

If $\eta_i = k \leq b_i$ the above probability coincides with the probability of visiting j_1 and j_2 consecutively, given that they have a demand. If $\eta_i > b_i$ the fact that customers j_1 and j_2 have demand is not a guarantee that they will be visited in the *a posteriori* solution. Also, it is possible that the arc (j_1, j_2) will be used even if customers between them have a demand since they may not be visited. In this case, the probability of using arc (j_1, j_2) given that its endpoints have a demand is expressed as a sum over customers t located between j_1 and j_2 and having a demand, i.e.,

$$p_k^+ = p^{k-2} q^{n_i - k} \sum_{t=t_0}^{t_1} \binom{\ell}{t} \binom{n_i - \ell - 2}{k - t - 2} \binom{k - t - 2}{b_i - 2} / \binom{k}{b_i}. \quad (28)$$

Summing p_k^- over all possible values k of η_i not higher than b_i and p_k^+ over all possible values k of η_i over b_i , yields the desired probability P_ℓ :

$$P_\ell = p^2 \left[\sum_{k=2}^{k_1} \binom{n_i - \ell - 2}{k - 2} p^{k-2} q^{n_i - k} + \sum_{k=b_i+1}^{n_i} p^{k-2} q^{n_i - k} \sum_{t=t_0}^{t_1} \binom{\ell}{t} \binom{n_i - \ell - 2}{k - t - 2} \binom{k - t - 2}{b_i - 2} / \binom{k}{b_i} \right],$$

where again $k_1 = \min\{b_i, n_i - \ell\}$, $t_0 = \max\{k - n_i + \ell, 0\}$, and $t_1 = \min\{k - b_i, \ell\}$.

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